

Math 260 Section 1.2  
Some possible results

Usually:

# equations = # unknowns  $\Rightarrow$  1 solution

# equations > # unknowns  $\Rightarrow$  0 solution

# equations < # unknowns  $\Rightarrow$   $\infty$  solutions

There are sometimes exceptions.

Situation	Typical number of solutions	Example	Row reduced augmented matrix	In equation form	So solution is	Solutions
# equations = # unknowns	1	$x + 2y = 5$ $3x + 4y = 6$	$\begin{bmatrix} 1 & 2 &   & 5 \\ 3 & 4 &   & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 &   & -4 \\ 0 & 1 &   & 9/2 \end{bmatrix}$	$x = -4$ $y = 9/2$	$x = -4$ $y = 9/2$	Only $(-4, 9/2)$
# equations > # unknowns	0	$x + 2y = 5$ $3x + 4y = 6$ $7x + 8y = 9$	$\begin{bmatrix} 1 & 2 &   & 5 \\ 3 & 4 &   & 6 \\ 7 & 8 &   & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 &   & -4 \\ 0 & 1 &   & 9/2 \\ 0 & 0 &   & 1 \end{bmatrix}$	$x = -4$ $y = 9/2$ $0 = 1$	No solution	None
# equations < # unknowns	$\infty$	$x + 2y + 3z = 6$ $4x + 5y + 6z = 12$	$\begin{bmatrix} 1 & 2 & 3 &   & 6 \\ 4 & 5 & 6 &   & 12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 &   & -2 \\ 0 & 1 & 2 &   & 4 \end{bmatrix}$	$x - z = -2$ $y + 2z = 4$	$x = -2 + z$ $y = 4 - 2z$ $z = z$	$(-2, 4, 0)$ or $(-1, 2, 1)$ or $(0, 0, 2) \dots$
# equations > # unknowns	0	$x + 2y = 5$ $3x + 4y = 6$ $7x + 8y = 8$	$\begin{bmatrix} 1 & 2 &   & 5 \\ 3 & 4 &   & 6 \\ 7 & 8 &   & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 &   & -4 \\ 0 & 1 &   & 9/2 \\ 0 & 0 &   & 0 \end{bmatrix}$	$x = -4$ $y = 9/2$ $0 = 0$	$x = -4$ $y = 9/2$	Only $(-4, 9/2)$
# equations = # unknowns	1	$x + 2y + 3z = 6$ $4x + 5y + 6z = 12$ $7x + 8y + 9z = 18$	$\begin{bmatrix} 1 & 2 & 3 &   & 6 \\ 4 & 5 & 6 &   & 12 \\ 7 & 8 & 9 &   & 18 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 &   & -2 \\ 0 & 1 & 2 &   & 4 \\ 0 & 0 & 0 &   & 0 \end{bmatrix}$	$x - z = -2$ $y + 2z = 4$ $0 = 0$	$x = -2 + z$ $y = 4 - 2z$ $z = z$	$(-2, 4, 0)$ or $(-1, 2, 1)$ or $(0, 0, 2) \dots$
# equations = # unknowns	1	$x + 2y + 3z = 6$ $4x + 5y + 6z = 12$ $7x + 8y + 9z = 19$	$\begin{bmatrix} 1 & 2 & 3 &   & 6 \\ 4 & 5 & 6 &   & 12 \\ 7 & 8 & 9 &   & 19 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 &   & -2 \\ 0 & 1 & 2 &   & 4 \\ 0 & 0 & 0 &   & 1 \end{bmatrix}$	$x - z = -2$ $y + 2z = 4$ $0 = 1$	No solution	None

Other possible results

Reduced row echelon matrix	Solutions (regardless of RHS)	Free variable in solution	Solution
$\begin{array}{ccc c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{array}$	1		$\begin{aligned} x_1 &= 2 \\ x_2 &= 3 \\ x_3 &= 4 \end{aligned}$
$\begin{array}{cccc c} 1 & 0 & 0 & 2 & 5 \\ 0 & 1 & 0 & 3 & 6 \\ 0 & 0 & 1 & 4 & 7 \end{array}$	$\infty$	$x_4$	$\begin{aligned} x_1 &= 5 - 2x_4 \\ x_2 &= 6 - 3x_4 \\ x_3 &= 7 - 4x_4 \\ x_4 &= x_4 \end{aligned}$
$\begin{array}{cccc c} 1 & 0 & 2 & 0 & 4 \\ 0 & 1 & 3 & 0 & 5 \\ 0 & 0 & 0 & 1 & 6 \end{array}$	$\infty$	$x_3$	$\begin{aligned} x_1 &= 4 - 2x_3 \\ x_2 &= 5 - 3x_3 \\ x_3 &= x_3 \\ x_4 &= 6 \end{aligned}$
$\begin{array}{ccccc c} 1 & 2 & 0 & 0 & 3 & 5 \\ 0 & 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 0 & 1 & 4 & 7 \end{array}$	$\infty$	$x_2, x_5$	$\begin{aligned} x_1 &= 5 - 2x_2 - 3x_5 \\ x_2 &= x_2 \\ x_3 &= 6 \\ x_4 &= 7 - 4x_5 \\ x_5 &= x_5 \end{aligned}$
$\begin{array}{cc c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & a \end{array}$	1 if $a = 0$ 0 if $a \neq 0$		$\begin{aligned} x_1 &= 2 \\ x_2 &= 3 \end{aligned}$
$\begin{array}{ccc c} 1 & 0 & 2 & 4 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & a \end{array}$	$\infty$ if $a = 0$ 0 if $a \neq 0$	$x_3$ if there is a solution	$\begin{aligned} x_1 &= 4 - 2x_3 \\ x_2 &= 5 - 3x_3 \\ x_3 &= x_3 \end{aligned}$
$\begin{array}{ccc c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & a \end{array}$	1 if $a = 0$ 0 if $a \neq 0$		$\begin{aligned} x_1 &= 2 \\ x_2 &= 3 \\ x_3 &= 4 \end{aligned}$
$\begin{array}{cccc c} 1 & 0 & 0 & 2 & 4 \\ 0 & 1 & 0 & 3 & 5 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 0 & a \end{array}$	$\infty$ if $a = 0$ 0 if $a \neq 0$	$x_4$ if there is a solution	$\begin{aligned} x_1 &= 4 - 2x_4 \\ x_2 &= 5 - 3x_4 \\ x_3 &= 6 \\ x_4 &= x_4 \end{aligned}$
$\begin{array}{ccc c} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & b \end{array}$	$\infty$ if $a = b = 0$ 0 otherwise	$x_2, x_3$ if there is a solution	$x_1 = 4 - 2x_2 - 3x_3$